4.3

Modulus and Conjugate

Learning Objectives:

- To define the modulus of a complex number and to study their properties

  AND

- To practice the related problems

The modulus of a complex number $z = x + iy$, denoted by $|z|$, is defined to be the nonnegative real number

$$|\boldsymbol{z}| = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2}$$

Example: Find the modulus of the complex number $z = 2 - 3i$

The modulus is given by

$$|z| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$$

The complex conjugate of the complex number $z = x + iy$ is given by $\bar{z} = x - iy$. Its modulus is given by

$$|\bar{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|$$

Both a complex number and its conjugate has the same modulus.

The product $z\bar{z}$ of a complex number and its conjugate is $|z|^2$.

$$z\bar{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2$$

The multiplicative inverse of a complex number $z \neq 0$ is denoted by z^{-1}, so that $zz^{-1} = 1$.

$$z^{-1} = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x}{x^2+y^2} + i\,\frac{-y}{x^2+y^2}$$

The multiplicative inverse can also be written as

$$\boldsymbol{z^{-1} = \frac{\bar{z}}{|z|^2}}$$

The modulus of product of two complex numbers is the product of their moduli.

$$|\boldsymbol{z_1 z_2}| = |\boldsymbol{z_1}||\boldsymbol{z_2}| \qquad \text{...... (1)}$$

This property also applies to division.

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad \text{............ (2)}$$

Example: Verify the property $|z_1 z_2| = |z_1||z_2|$ holds for the product of $z_1 = 3 + 2i$ and $z_2 = -1 - 4i$.

Solution:

$$|z_1 z_2| = |(3 + 2i)(-1 - 4i)| = |5 - 14i|$$

$$= \sqrt{(5)^2 + (-14)^2} = \sqrt{221}$$

$$|z_1| = \sqrt{(3)^2 + (2)^2} = \sqrt{13}$$

$$|z_2| = \sqrt{(-1)^2 + (-4)^2} = \sqrt{17}$$

Hence $\qquad |z_1||z_2| = \sqrt{13}\sqrt{17} = \sqrt{221} = |z_1 z_2|$

P1:

Express $(1-i)^4$ in the form of $x+iy$.

Solution:

We have,

$$(1-i)^4 = (1-i)^2(1-i)^2 = (1-2i+i^2)(1-2i+i^2)$$

$$= (-2i)(-2i) = 4i^2 = -4$$

P2:

Find $\left|\frac{1}{(2+i)^2}-\frac{1}{(2-i)^2}\right|$.

Solution:

$$\left|\frac{1}{(2+i)^2}-\frac{1}{(2-i)^2}\right| = \left|\frac{(2-i)^2-(2+i)^2}{[(2+i)(2-i)]^2}\right| = \left|\frac{-8i}{(4+1)^2}\right| = \frac{8}{25}$$

P3:

Show that $z_1 = \frac{2+11i}{25}$ and $z_2 = \frac{-2+i}{(1-2i)^2}$ are conjugate to each other.

Solution:

We have, $z_1 = \frac{2+11i}{25}, z_2 = \frac{-2+i}{(1-2i)^2}$

$$z_2 = \frac{-2+i}{(1-2i)^2} = \frac{-2+i}{1+4i^2-4i} = \frac{-2+i}{1-4-4i}$$

$$= \frac{-2+i}{-3-4i} \times \frac{-3+4i}{-3+4i} = \frac{6-8i-3i+4i^2}{9-16i^2}$$

$$= \frac{6-8i-3i-4}{9+16} = \frac{2-11i}{25} = \overline{z_1}$$

Therefore, z_1 and z_2 are conjugate to each other.

P4:

If $z = \frac{1}{1+\cos\theta + i\sin\theta}$, then find $|z|^2$.

Solution:

We have, $z = \dfrac{1}{(1+\cos\theta)+i\sin\theta} \times \dfrac{(1+\cos\theta)-i\sin\theta}{(1+\cos\theta)-i\sin\theta}$

$$\Rightarrow z = \frac{(1+\cos\theta)-i\sin\theta}{(1+\cos\theta)^2-i^2\sin^2\theta} = \frac{(1+\cos\theta)-i\sin\theta}{1+\cos^2\theta+2\cos\theta+\sin^2\theta}$$

$$\Rightarrow z = \frac{(1+\cos\theta)-i\sin\theta}{1+\cos^2\theta+\sin^2\theta+2\cos\theta} = \frac{(1+\cos\theta)-i\sin\theta}{1+1+2\cos\theta}$$

$$\Rightarrow z = \frac{(1+\cos\theta)-i\sin\theta}{2(1+\cos\theta)} = \frac{(1+\cos\theta)}{2(1+\cos\theta)} - \frac{i\sin\theta}{2(1+\cos\theta)}$$

$$\Rightarrow z = \frac{1}{2} - \frac{\sin\theta}{2(1+\cos\theta)}i$$

$$\therefore |z|^2 = \frac{1}{4} + \frac{\sin^2\theta}{4(1+\cos\theta)^2} = \frac{1}{4}\left(\frac{1+\cos^2\theta+2\cos\theta+\sin^2\theta}{(1+\cos\theta)^2}\right)$$

$$= \frac{1}{4}\left(\frac{2(1+\cos\theta)}{(1+\cos\theta)^2}\right) = \frac{1}{2(1+\cos\theta)}$$

IP1:

Express $\frac{2+3i}{(7-i)(4+2i)}$ in the form of $x+iy$.

Solution:

We have, $\frac{2+3i}{(7-i)(4+2i)}=\frac{2+3i}{28+10i-2i^2}=\frac{2+3i}{30+10i}$

$$=\frac{1}{10}\left(\frac{2+3i}{3+i}\right)=\frac{1}{10}\left(\frac{2+3i}{3+i}\times\frac{3-i}{3-i}\right)$$

$$=\frac{1}{10}\left(\frac{6+7i-3i^2}{9-i^2}\right)=\frac{1}{10}\left(\frac{9+7i}{10}\right)$$

$$=\frac{9+7i}{100}$$

IP2:

If $z_1 = 2 - i, z_2 = 1 + i$, find $\left|\frac{z_1+z_2+1}{z_1-z_2+1}\right|$.

Solution:

We have, $z_1 = 2 - i, z_2 = 1 + i$

$$\therefore \left|\frac{z_1+z_2+1}{z_1-z_2+1}\right| = \left|\frac{2-i+1+i+1}{2-i-1-i+1}\right| = \left|\frac{4}{2-2i}\right| = \left|\frac{2}{1-i}\right| = \frac{|2|}{|1-i|} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

IP3:

Find the complex conjugate of $\frac{2-i}{(1-2i)^2}$.

Solution:

$$\frac{2-i}{(1-2i)^2} = \frac{2-i}{1+4i^2-4i}$$

$$= \frac{2-i}{1-4-4i} = \frac{2-i}{-3-4i} \times \frac{-3+4i}{-3+4i} = \frac{-6+11i-4i^2}{9-16i^2}$$

$$= \frac{-6+11i+4}{9+16} = \frac{-2+11i}{25} = -\frac{2}{25} + \frac{11}{25}i$$

Therefore,

the conjugate of $\frac{2-i}{(1-2i)^2}$ $=$ the conjugate of $-\frac{2}{25} + \frac{11}{25}i$

$$= -\frac{2}{25} - \frac{11}{25}i$$

IP4:

Find the multiplicative inverse of $\frac{\sqrt{5}+3i}{3+\sqrt{5}i}$.

Solution:

Let $z = \frac{\sqrt{5}+3i}{3+\sqrt{5}i} = \frac{\sqrt{5}+3i}{3+\sqrt{5}i} \times \frac{3-\sqrt{5}i}{3-\sqrt{5}i}$

$$= \frac{3\sqrt{5}-5i+9i-3\sqrt{5}i^2}{9-5i^2} = \frac{3\sqrt{5}+4i+3\sqrt{5}}{9+5}$$

$$= \frac{6\sqrt{5}+4i}{14} = \frac{3\sqrt{5}+2i}{7}$$

$$\therefore z = \frac{3\sqrt{5}+2i}{7}$$

Now, $z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{\frac{3\sqrt{5}-2i}{7}}{\sqrt{\frac{45}{49}+\frac{4}{49}}} = \frac{\frac{3\sqrt{5}-2i}{7}}{\sqrt{\frac{49}{49}}} = \frac{3\sqrt{5}-2i}{7}$

1. If $z = x + iy$, then prove

$$\frac{z}{\bar{z}} = \left(\frac{x^2 - y^2}{x^2 + y^2}\right) + i\left(\frac{2xy}{x^2 + y^2}\right)$$

2. Express $(5-3i)^3$ in the form $x+iy$.

3. Find the multiplicative inverse of $2 - 3i$.

4. Express $\frac{5+i\sqrt{2}}{1-i\sqrt{2}}$ in the form $x + iy$

4.4

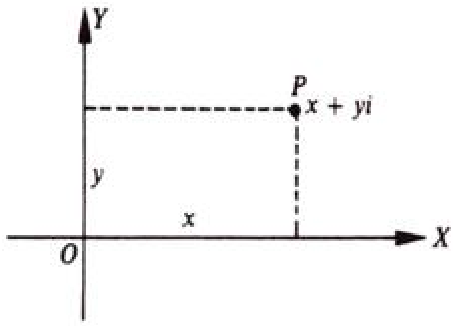
Complex Plane

Learning Objectives:

- To study the representation of complex numbers
- To define the argument of a complex number and to study its properties

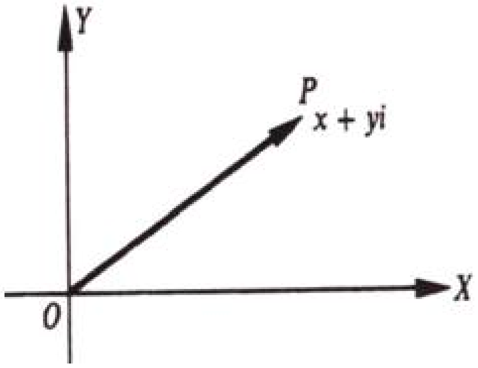
  AND
- To practice the related problems

The complex number $x + iy$ may be represented graphically by the point P whose rectangular coordinates are (x, y).

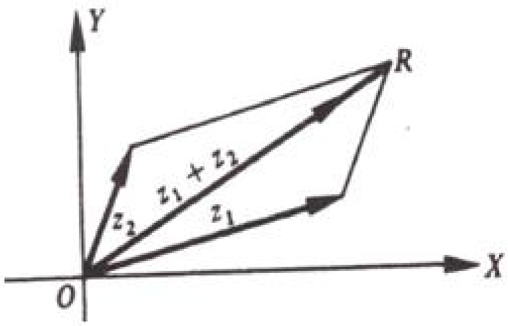


The point O, having the coordinates $(0,0)$, represents the complex number $0 + 0i = 0$. All points on the x-axis have coordinates of the form $(x, 0)$ and correspond to real numbers $x + 0i = x$. For this reason, the x-axis is called the *axis of reals* or *real axis*. All points on the y-axis have coordinates of the form $(0, y)$ and correspond to imaginary numbers $0 + iy = iy$. The y-axis is called the *axis of imaginaries* or *imaginary axis*. The plane on which the complex numbers are assigned to each of its points is called the *complex plane*.

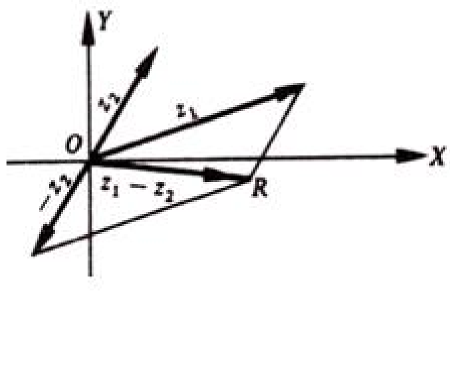
In addition to representing a complex number by a point P on the complex plane, the number may be represented by the directed line segment OP. The directed line segment OP is also called *vector OP.*



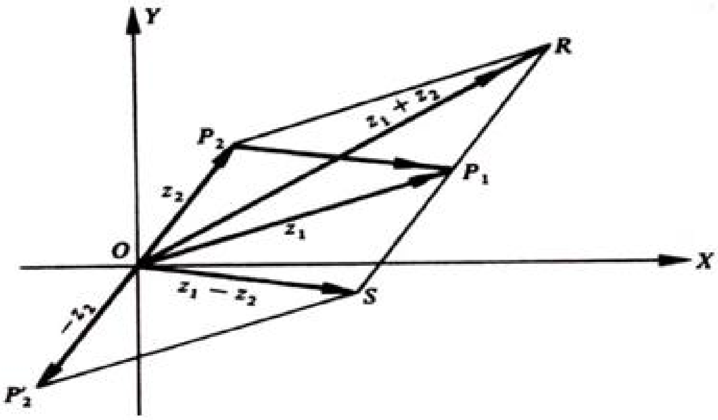
Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers. We add these two numbers by using the parallelogram law of vectors.



We can also obtain the difference $z_1 - z_2$ of the two complex numbers by applying the law of parallelogram to z_1 and $-z_2$.



In the figure below, the vector OR gives the sum $z_1 + z_2$ and the vector OS gives the difference $z_1 - z_2$.



It is noted that the segments OS and P_2P_1 are equal. P_2P_1 is the other diagonal of the parallelogram OP_2RP_1.

Modulus and Argument

The modulus of the complex number $z = x + iy$ is $\sqrt{x^2 + y^2}$ and it is the distance of the corresponding point from the origin in the complex plane.

The argument of the complex number z is denoted by $arg\ z$ and is defined as

$$\mathbf{arg\ z = tan^{-1}\left(\frac{y}{x}\right)}$$

It can be seen that $\arg z$ is the angle that the line joining the origin to z on the complex plane makes with the positive x-axis. The anticlockwise direction is taken to be positive by convention.

Example: Find the argument of the complex number $z = 2 - 3i$.

The argument is given by

$$\arg z = \tan^{-1}\left(-\frac{3}{2}\right)$$

The two angles whose tangent equal -1.5 are in second and fourth quadrants.

The argument of the product of two complex numbers is equal to the sum of the arguments of the complex numbers.

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \qquad \text{... (1)}$$

Similarly, the argument of quotient of two complex numbers is equal to the difference of their arguments.

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

We now examine the effect on a complex number z of multiplying it by ± 1 and by $\pm i$. These four multipliers have modulus unity. From property $|z_1 z_2| = |z_1||z_2|$, it follows that multiplying z by another complex number of unit modulus gives a product with the same modulus as z. We can also see from equation (1) that if we multiply z by a complex number then the argument of the product is the sum of the argument of z and the argument of the multiplier.

Hence multiplying z by unity (which has argument zero) leaves z unchanged in both modulus and argument, that is, z is completely unaltered by the operation. Multiplying z by −1 (which has argument π) leads to rotation, through an angle of π, of the line joining the origin to z in the complex plane. Similarly, multiplication by i or $-i$ leads to corresponding rotations of π/2 or −π/2 respectively.

Example: Using the geometrical interpretation multiplication by i, find the product of $i(1-i)$.

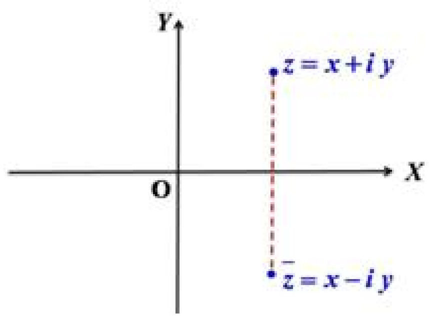
The complex number $1-i$ has argument −π/4 and modulus $\sqrt{2}$. Its product with i has argument +π/4 and unchanged modulus $\sqrt{2}$. The complex number with modulus $\sqrt{2}$ and argument +π/4 is $1+i$ and so

$$i(1-i) = 1+i$$

Of course, this can be easily verified by direct multiplication.

Complex Conjugate

The complex conjugate of a complex number $z = x + iy$ is $\bar{z} = x - iy$. Complex conjugation corresponds to a reflection of z in the real axis of the complex plane, as shown below.



P1:

Find the modulus and argument of the complex number $\frac{1}{1+i}$

Solution:

The modulus of $\frac{1}{1+i}$ is $\left|\frac{1}{1+i}\right| = \frac{|1|}{|1+i|} = \frac{1}{\sqrt{2}}$

Given, $\frac{1}{1+i} = \frac{1}{1+i} \times \frac{1-i}{1-i} = \frac{1-i}{1-i^2} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i$

Since the argument of $x + iy$ is $\tan^{-1}\left(\frac{y}{x}\right)$, the argument of $\frac{1}{2} - \frac{1}{2}i$ is $\tan^{-1}(-1) = -\tan^{-1} 1 = -\frac{\pi}{4}$

P2:

If z is a non-zero complex number, then find $\arg \bar{z}$.

Solution:

Given, z is a non-zero complex number.

Then $z\bar{z} = |z|^2 \neq 0$ and $|z|^2$ is positive real number.

Now, $\arg z + \arg \bar{z} = \arg z\bar{z} = \arg|z|^2 = \tan^{-1}\left(\frac{0}{x^2+y^2}\right) = 0$

$$\therefore \mathbf{\arg \bar{z} = -\arg z}$$

Note:

If z is a non-zero complex number, then $\mathbf{\arg z + \arg \bar{z} = 0}$.

P3:

If $\frac{\pi}{2}$ and $\frac{\pi}{4}$ are respectively the arguments of z_1 and $\overline{z_2}$, then find the value of $\arg\left(\frac{z_1}{z_2}\right)$.

Solution:

Given, $\arg(z_1) = \frac{\pi}{2}$ and $\arg(\overline{z_2}) = \frac{\pi}{4} \Rightarrow \arg(z_2) = -\frac{\pi}{4}$

$$\therefore \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

P4:

Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

Solution:

We have, $(x - iy)(3 + 5i) = 3x + 5xi - 3yi - 5yi^2$

$$= 3x + 5xi - 3yi + 5y$$

$$= (3x + 5y) + (5x - 3y)i$$

It is given that $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

$$\therefore (x - iy)(3 + 5i) = \overline{(-6 - 24i)}$$

$$\Rightarrow (3x + 5y) + (5x - 3y)i = -6 + 24i$$

Equating real and imaginary parts, we get

$$\Rightarrow 3x + 5y = -6 \text{ and } 5x - 3y = 24$$

Solving the above equations, we get $x = 3$ and $y = -3$.

IP1:

Find the modulus and argument of the complex number $\frac{1+i}{1-i}$

Solution:

Given, $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{1+i^2+2i}{1-i^2} = \frac{2i}{2} = i$

Therefore the modulus of i is 1.

Since the argument of $x + iy$ is $\tan^{-1}\left(\frac{y}{x}\right)$, the argument of i is $\tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$

IP2:

If $\frac{\pi}{5}$ and $\frac{\pi}{3}$ are the arguments of $\overline{z_1}$ and z_2, then find the value of $\arg(z_1 z_2)$.

Solution:

Given, $\arg(\overline{z_1}) = \frac{\pi}{5}$ and $\arg(z_2) = \frac{\pi}{3}$

Let $z_1 = x_1 + iy_1$

$\therefore \overline{z_1} = x_1 - iy_1$

$\because$ Argument of $\overline{z_1} = \tan^{-1}\left(-\frac{y_1}{x_1}\right)$

Given, $\tan^{-1}\left(-\frac{y_1}{x_1}\right) = \frac{\pi}{5} \Rightarrow -\tan^{-1}\left(\frac{y_1}{x_1}\right) = \frac{\pi}{5}$

$$\Rightarrow \tan^{-1}\left(\frac{y_1}{x_1}\right) = -\frac{\pi}{5} \Rightarrow \arg(z_1) = -\frac{\pi}{5}$$

$$\therefore \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = -\frac{\pi}{5} + \frac{\pi}{3} = \frac{-3\pi+5\pi}{15} = \frac{2\pi}{15}$$

IP3:

If $z_1 = -1$ and $z_2 = i$, find the value of $\arg\left(\frac{z_1}{z_2}\right)$.

Solution:

We have, $z_1 = -1$ and $z_2 = i$

$$\therefore \arg\left(\frac{z_1}{z_2}\right) = \arg\left(\frac{-1}{i}\right) = \arg(i) = \frac{\pi}{2}$$

IP4:

Find real values of x and y for which the complex numbers $-3+ix^2y$ and x^2+y+4i are conjugate of each other.

Solution:

Since $-3+ix^2y$ is the conjugate of x^2+y+4i,

$$-3+ix^2y=\overline{(x^2+y+4i)}$$

$$\Rightarrow -3+ix^2y=(x^2+y)-4i$$

Equating real and imaginary parts, we get

$$\Rightarrow -3=x^2+y \text{ and } x^2y=-4$$

$$\Rightarrow -3=x^2+y \text{ ---------------------- (1)}$$

$$\text{and } x^2y=-4 \Rightarrow y=-\frac{4}{x^2} \text{ ----------- (2)}$$

From (1) and (2), we get $-3=x^2-\frac{4}{x^2}$

$$\Rightarrow x^4+3x^2-4=0 \Rightarrow (x^2+4)(x^2-1)=0$$

$$\Rightarrow x^2+4=0 \text{ or } x^2-1=0$$

$$\Rightarrow x=\pm 2i \text{ or } x=\pm 1$$

$$\Rightarrow x=\pm 1 \quad (\text{ since } x \text{ is a real number }) \text{ ------ (3)}$$

From (2) and (3), we get $y=-4$.

Hence, $x=-1, y=-4$ or $x=1, y=-4$.

1. Prove that

a. $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

b. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$